MATH 220A

Midterm Exam, November 8, 2019

Instructions: 3 hours. You may use without proof results proved in Conway up to and including Chapter IV. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

1. (15+15=30p) Suppose that the radius of convergence (ROC) of the power series $\sum_{n=0}^{\infty} a_n z^n$ is $R > 1$.

(**a**) Show that $\lim\limits_{n\to\infty} a_n = 0$. (**b**) Compute ROC for $\displaystyle\sum_{n=0}^{\infty}(2+a_n)^{n^2} a_n z^{n^2}$.

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2. (30p) Let γ be the closed curve given by the ellipse $x^2 + \frac{y^2}{4} = 1$ traversed once in the positive (counterclockwise) direction. Compute

$$\int_\gamma \frac{e^{iz^3}}{z^2+1} dz.$$

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3. (30p) Let γ be the half-circle $\gamma(t) = e^{it}$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Compute

$$\int_\gamma z e^{iz} dz.$$

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4. (30p) For $a \in (-1, 1)$, let $D_a = \{z \colon |z| < 1,\ \text{Re } z > a\}$. For each such a, either find a Möbius transformation of D_a onto the quadrant $Q = \{w = re^{i\theta} \colon r > 0,\ 0 < \theta < \frac{\pi}{2}\}$, or show that such cannot exist.

5. (15+15=30p) Let f be analytic in the open disk $B(0,2)$ and continuous in the closed disk $\overline{B(0,2)}$.

(a) Show that $\sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}$ converges uniformly in the closed disk $\overline{B(0,c)}$ for every $c < 1$.

(b) Show that the function defined by this series is analytic in the unit disk $B(0,1)$.

Note: A result from a homework problem is relevant to (b). If you use it, you must reprove the result.

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6. (30p) Let f be analytic in the open disk $B(0, 1+\delta)$ with $\delta > 0$. Assume that $|f(z)| = 1$ on the unit circle $|z| = 1$ and that f has m zeros, counted with multiplicities, in the unit disk $\mathbb{D} = B(0, 1)$. Show that the equation $f(z) = \alpha$ has exactly m roots, counted with multiplicities, for every $\alpha \in \mathbb{D}$.
